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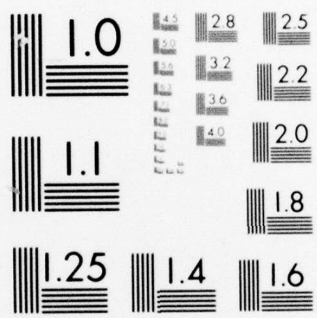
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BY

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1. Introduction

The general quadratic programming problem is

$$\begin{aligned} &\text{minimize:} \quad \frac{1}{2} x^T D x + c^T x \\ &\text{subject to:} \quad Ax \leq b \end{aligned} \tag{1}$$

In [5], Eaves describes a procedure that in a finite number of steps, determines either the global minimum of (1) or a halfline of the constraint set along which the minimand is unbounded below. That a quadratic function on any nonempty polyhedral convex set either attains its infinum there or is unbounded below on a halfline of the set is given to us by the Frank-Wolfe theorem, [4,6].

In this paper we shall present an alternative method that also accomplishes this task in a finite number of steps. Like that of Eaves, it is mainly of theoretical interest, being computationally useful only on problems that have a small number of variables. Our approach will be to adapt the ideas in [7] to obtain a recursive procedure, that is, one which begins with a problem in n variables, and then reduces it to a problem of exactly the same form in fewer than n variables.

Other finite methods for finding global minima of quadratic programs (see for example [2]) do so by obtaining the best local minimum and are unable to detect whether or not the constrained minimand is unbounded below.

2. Notation

Let $\underline{n}$ denote $\{1, 2, \ldots, n\}$. For $x \in R^n$, $\alpha \subseteq \underline{n}$, let $x_\alpha \in R^k$ denote $(x_{\alpha_1}, \ldots, x_{\alpha_k})^T$ where $\alpha = \{\alpha_1, \ldots, \alpha_k\}$, $\alpha_1 < \cdots < \alpha_k$.

For $A \in R^{m \times n}$, $\alpha \subseteq \underline{m}$, $\beta \subseteq \underline{n}$, let $A_{\alpha .}$ denote the submatrix of A whose rows are indexed by α; let $A_{. \beta}$ denote the submatrix of A whose columns are indexed by β; let $A_{\alpha\beta}$ denote $(A_{\alpha .})_{. \beta}$.

Let (D, c, A, b) denote the quadratic program in (1) where $D \in R^{n \times n}$, $c \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$.

Let $\boxed{X}$ denote the end of the procedure.

3. Preliminary Results

We require the following elementary and well known results which we shall state without proof (see for example [8]).

Let $P(b)$ be a polyhedral convex set of the form

$$P(b) = \{x \in R^n : Ax \leq b\}$$

where $A \in R^{m \times n}$. In particular

$$P(0) = \{x \in R^n : Ax \leq 0\} .$$

<u>3.1 Lemma</u>: If $P(b)$ is nonempty then $P(b)$ is bounded iff $P(0) = \{0\}$.

<u>3.2 Lemma</u>: If $P(b)$ is nonempty, and $f: R^n \to R$ is a strictly concave function, then

(i) f is unbounded below on $P(b)$ iff $P(0) \neq \{0\}$; further, if $0 \neq y \in P(0)$ and $x \in P(b)$, then $f(x + \theta y) \to -\infty$ as $\theta \to \infty$

(ii) if $P(0) = \{0\}$ then f attains its infimum on $P(b)$; moreover this infimum is attained only at an extreme point of $P(b)$.

<u>3.3 Lemma</u>: Let $x \in P(b)$, and let $\gamma = \{i : A_{i.} x = b_i\}$. Then x is an extreme point of $P(b)$ iff rank $(A_{\gamma.}) = n$, i.e., $A_{\gamma.}$ has full column rank.

From Lemma 3.3 we obtain immediately

<u>3.4 Lemma</u>: There is a finite collection $\mathcal{M}$ of subsets of <u>m</u> such that for all b, x is an extreme point of $P(b)$ iff there is a $\gamma \in \mathcal{M}$ such that

(i) $A_{\gamma.} x = b_\gamma$

(ii) $A_{\gamma.}^{-1}$ exists

(iii) $A_{\delta.}x \leq b_{\delta}$ where $\delta = \underline{m} \sim \gamma$

In addition to these lemmas we shall require a finite procedure for the convex case, that is, when the matrix D in (1) is positive semi-definite. There are several algorithms [3] available for this task, and so, in what follows we shall simply assume that the convex case can be (relatively) easily solved in a finite number of steps.

4. <u>The Procedure</u>

We shall assume that the constraint set $\{x : Ax \leq b\}$ is non-empty, and without loss of generality that D is symmetric.

We now proceed as follows:

<u>4.1</u>: The case $n = 0$. Here the problem is vacuous. Define the optimal value to be zero and attained at the origin $\boxed{X}$.

<u>4.2</u>: Since D is symmetric, there exists an orthogonal matrix Q [1, p. 54] such that

$$Q^T DQ = \text{Diag}\ (\lambda_1, \ldots, \lambda_n)$$

where the λ_i are the real eigenvalues of D. Setting $\bar{D} = \text{Diag}\ (\lambda_1, \ldots, \lambda_n)$, $\bar{c} = Q^T c$ and $\bar{A} = AQ$ we obtain an equivalent quadratic program $(\bar{D}, \bar{c}, \bar{A}, b)$

in the transformed variables $\bar{x}$, where $\bar{x} = Q^T x$. Let $\alpha = \{i:\lambda_i \geq 0\}$ and $\beta = \{i:\lambda_i < 0\}$.

4.3: The convex case, that is, when $\beta = \phi$. By our remarks in Section 3, we may assume this done $\boxed{X}$.

4.4: The non-convex case $(\beta \neq \phi)$ is the interesting case. Since $\bar{D}$ is diagonal, we may write the problem $(\bar{D}, \bar{c}, \bar{A}, b)$ as

$$\begin{aligned} &\text{minimize:} && \left(\frac{1}{2}\bar{x}_\alpha^T \bar{D}_{\alpha\alpha}\bar{x}_\alpha + \bar{c}_\alpha^T \bar{x}_\alpha\right) + \left(\frac{1}{2}\bar{x}_\beta \bar{D}_{\beta\beta}\bar{x}_\beta + \bar{c}_\beta^T \bar{x}_\beta\right) \\ &\text{subject to:} && \bar{A}_{\cdot\alpha}\bar{x}_\alpha + \bar{A}_{\cdot\beta}\bar{x}_\beta \leq b \end{aligned} \tag{2}$$

The second term in the minimand is strictly concave since $\lambda_i < 0$ for $i \in \beta$. By Lemma 3.2(i) the minimand is unbounded below if there exists $\bar{y}_\beta \neq 0$ such that $\bar{A}_{.\beta}\bar{y}_\beta \leq 0$. Whether or not such a $\bar{y}_\beta$ exists can be determined by solving a linear program.

If such a $\bar{y}_\beta$ does exist, the minimand in (2) is unbounded below on the halfline $\{\bar{x} + \theta\bar{y}:\theta \geq 0\}$ where $\bar{y} = (0\ \bar{y}_\beta)$ and $\bar{x}$ satisfies $\bar{A}\bar{x} \leq b$. Therefore the minimand of the original problem (D, c, A, b) is unbounded below on the halfline $\{x + \theta y:\theta \geq 0\}$ where $x = Q\bar{x}$ and $y = Q\bar{y}$.

If no such $\bar{y}_\beta$ exists, then for all $\bar{x}_\alpha$, the set

$$R(\bar{x}_\alpha) \triangleq \{\bar{x}_\beta:\bar{A}_{.\beta}\bar{x}_\beta \leq b - \bar{A}_{.\alpha}\bar{x}_\alpha\}$$

is bounded. This follows from Lemma 3.1. By Lemma 3.4 there is a finite collection $\mathcal{M}$ of subsets of $\underline{m}$ such that for each $\bar{x}_\alpha$, $\bar{x}_\beta$ is an extreme point of $R(\bar{x}_\alpha)$ iff there is a $\gamma \in \mathcal{M}$ such that

$$\bar{A}_{\gamma\beta}\bar{x}_\beta = b_\gamma - \bar{A}_{\gamma\alpha}\bar{x}_\alpha ,$$

$$\bar{A}^{-1}_{\gamma\beta} \text{ exists,} \tag{3}$$

and

$$A_{\delta\beta}\bar{x}_\beta \leq b_\delta - \bar{A}_{\delta\alpha}\bar{x}_\alpha \qquad \text{where } \delta = \underline{m} \sim \gamma . \tag{4}$$

Further, since the constraint set in (2) is nonempty, there is an $\bar{x}'_\alpha$ such that $R(\bar{x}'_\alpha)$ is nonempty. Since $R(\bar{x}'_\alpha)$ is bounded it contains extreme points and hence $\mathcal{M}$ is nonempty.

Now, for each fixed $\gamma \in \mathcal{M}$, use (3) to eliminate the variables $\bar{x}_\beta$ in (2). We do this by writing

$$\bar{x}_\beta = H^\gamma \bar{x}_\alpha + h^\gamma \tag{5}$$

where

$$H^\gamma = -\bar{A}^{-1}_{\gamma\beta}\bar{A}_{\gamma\alpha} , \qquad h^\gamma = \bar{A}^{-1}_{\gamma\beta} b_\gamma$$

and substituting for $\bar{x}_\beta$ in the remaining constraints (4) and the minimand in (2). This yields a reduced problem $(D^\gamma, c^\gamma, A^\gamma, b^\gamma)$ in the variables $\bar{x}_\alpha$, where

$$D^\gamma = \bar{D}_{\alpha\alpha} + (H^\gamma)^T \bar{D}_{\beta\beta} H^\gamma$$

$$c^\gamma = \bar{c}_\alpha + (H^\gamma)^T (\bar{c}_\beta + \bar{D}_{\beta\beta} h^\gamma)$$

$$A^\gamma = \bar{A}_{\delta\alpha} + \bar{A}_{\delta\beta} H^\gamma$$

$$b^\gamma = b_\delta - \bar{A}_{\delta\beta} h^\gamma \; .$$

Note that by substituting (5) into the minimand of (2) we also obtain a constant term

$$g^\gamma = \frac{1}{2} (h^\gamma)^T \bar{D}_{\beta\beta} h^\gamma + \bar{c}_\beta^T h^\gamma \; .$$

We now apply the recursion to each of the smaller problems $(D^\gamma, c^\gamma, A^\gamma, b^\gamma)$, that is, we go back to step 4.1 of this procedure with (D, c, A, b) replaced by $(D^\gamma, c^\gamma, A^\gamma, b^\gamma)$. Eventually we must terminate in a finite number of steps at either the case $n = 0$ or the convex case.

There are now two possibilities:

(i) For some γ, the minimand in $(D^\gamma, c^\gamma, A^\gamma, b^\gamma)$ is unbounded below on a halfline $\{u^\gamma + \theta v^\gamma : \theta \geq 0\}$. Using the relation (5) and imbedding this halfline back in the constraint set of (2), it follows that

$$\left\{\begin{pmatrix} u^\gamma \\ H^\gamma u^\gamma + h^\gamma \end{pmatrix} + \theta \begin{pmatrix} v^\gamma \\ H^\gamma v^\gamma \end{pmatrix} : \theta \geq 0 \right\}$$

is a halfline of this constraint set on which the minimand in (2) is unbounded below. Transforming this ray appropriately using the matrix Q yields the corresponding ray of the original set in (1) along which that minimand is unbounded below.

(ii) For each γ, the problem $(D^\gamma, c^\gamma, A^\gamma, b^\gamma)$ has a global minimand W^γ at some point u^γ.

Set

$$\rho = \min_{\gamma \in \mathcal{M}} \{W^\gamma + g^\gamma\} .$$

and let

$$\gamma^* = \operatorname{argmin}_{\gamma \in \mathcal{M}} \{W^\gamma + g^\gamma\}$$

so that $\rho = W^{\gamma^*} + g^{\gamma^*}$.

Then ρ is the global minimum of the problem $(\bar{D}, \bar{c}, \bar{A}, b)$ and is attained at the point

$$\bar{x}^* = \begin{pmatrix} u^{\gamma^*} \\ H^{\gamma^*} u^{\gamma^*} + h^{\gamma^*} \end{pmatrix}$$

This is readily seen to be a consequence of Lemma 3.2(ii). Transforming back as before yields ρ the global minimum of the problem (D, c, A, b) at the point x^* where $x^* = Q\bar{x}^*$ $\boxed{X}$.

This completes the procedure.

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This paper considers the global analysis of general quadratic programs in a finite number of steps. A procedure is presented for recursively finding either the global minimum or a halfline of the constraint set along which the minimand is unbounded below.

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